

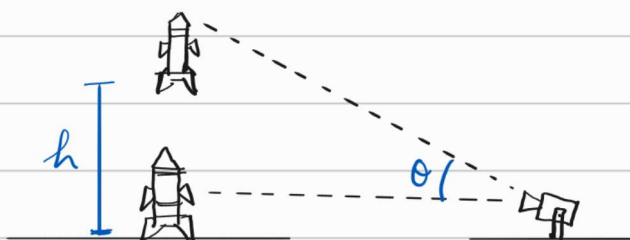
Calculus I - Spr 26

Lecture 23

Mar 6

Sec 3.9: Related Rates

If you have watched rocket launches, you would have noticed that the cameras track the rocket and always keep it in frame.



Let $h(t)$ = height of rocket from ground at time t .
and $\theta(t)$ = angle of elevation of the camera from horizontal level.

Clearly h and θ are related. As h increases so does θ .

When you have two such quantities which are related, their rates are also related.

h and θ related $\Rightarrow \frac{dh}{dt}$ and $\frac{d\theta}{dt}$ related.

Most times it is easier to compute / know one.

We can use the relation between them to compute the other.

This is the idea behind related rates.

Eg. Suppose the distance of the rocket from the ground increases at a rate of 50 m/s. At what rate should you be tilting your camera up when the angle $\theta = 30^\circ$ and when $\theta = 60^\circ$.

Assume the camera is 200m away from the launch pad.

Soln. We start by identifying two things:

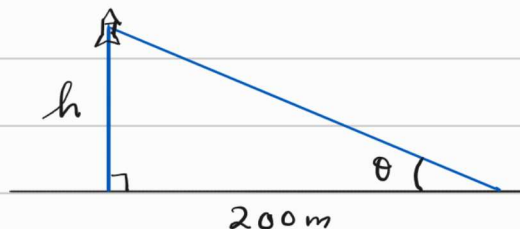
- the given information:
the rate of increase of the distance from ground.
- the unknown:
the rate of increase of the angle b/w the camera and the ground when $\theta = 30^\circ, 60^\circ$.

Suggestive Notation: Let h be the distance from ground and θ denote the camera angle.

Given: $\frac{dh}{dt} = 50 \text{ m/s}$

Unknown: $\frac{d\theta}{dt}$ when $\theta = 30^\circ, 60^\circ$

To find a relation b/w the rates of change of two quantities, we first find a relation b/w the quantities.
(a key step)



$$\tan \theta = \frac{h}{200} \quad (\text{relation b/w } \theta \text{ and } h)$$

$$\Leftrightarrow h = 200 \tan \theta$$

Once you have a relation, differentiate both sides wrt t .

$$\frac{d(h)}{dt} = \frac{d}{dt} (200 \tan \theta)$$

$$\frac{dh}{dt} = 200 \frac{d}{dt} (\tan \theta)$$

$$= 200 \frac{d}{d\theta} \tan \theta \cdot \frac{d\theta}{dt}$$

$$= 200 \sec^2 \theta \frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{200 \sec^2 \theta} \frac{dh}{dt}$$

$$= \frac{\cos^2 \theta}{200} \frac{dh}{dt}$$

We want to know $\frac{d\theta}{dt}$ when $\theta = 30^\circ$ and $\theta = 60^\circ$.

$$\text{i) } \theta = 30^\circ \quad \frac{d\theta}{dt} = \frac{\cos^2(\frac{\pi}{6}) \text{ rad}}{200 \text{ m}} \cdot 50 \text{ m/s}$$

$$(\theta = \frac{\pi}{6} \text{ rad})$$

$$= \frac{(\frac{3}{4})}{200} \cdot 50 \text{ rad/s}$$

$$= \frac{3}{16} \text{ rad/s}$$

$$\approx 0.19 \text{ rad/s}$$

$$\text{ii) } \theta = 60^\circ \quad \frac{d\theta}{dt} = \frac{\cos^2(\frac{\pi}{3}) \text{ rad}}{200 \text{ m}} \cdot 50 \text{ m/s}$$

$$(\theta = \frac{\pi}{3} \text{ rad})$$

$$= \frac{\left(\frac{1}{4}\right)}{200} \cdot 50 \text{ rad/s}$$

$$= \frac{1}{16} \text{ rad/s}$$

$$\approx 0.06 \text{ rad/s}$$

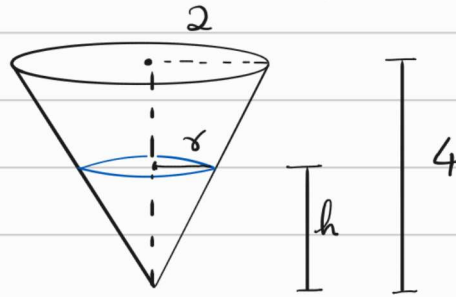
At a higher angle, you only need to move it slower to keep the rocket in frame.

Problem Solving Strategy:

- 1) Read the problem carefully.
- 2) Draw a diagram if possible.
- 3) Introduce notation. Assign symbols to all quantities that are functions of time.
- 4) Express the given information and the required rate in terms of derivatives.
- 5) Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
- 6) Use the chain rule to differentiate both sides of the equation with respect to t .
- 7) Substitute the given information into the resulting equation and solve for the unknown rate.

Eg. A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2\text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3m deep.

Soln.



V = Volume of water

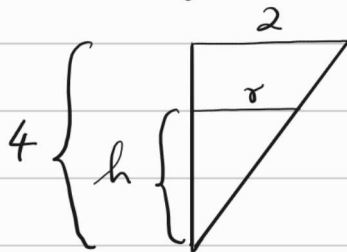
h = Height of water level

r = Radius of the surface

Given $\frac{dV}{dt} = 2\text{m}^3/\text{min}$

Unknown $\frac{dh}{dt}$ when $h = 3\text{m}$.

Volume of the cone  $V = \frac{1}{3}\pi r^2 h$



By similar triangles

$$\frac{r}{h} = \frac{2}{4}$$

$$\Rightarrow r = \frac{h}{2}$$

$$\text{So, } V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

$$\Rightarrow \frac{dV}{dt} = \frac{d}{dt} \left(\frac{1}{12}\pi h^3 \right)$$

$$= \frac{\pi}{12} \frac{d}{dh}(h^3) \cdot \frac{dh}{dt}$$

$$= \frac{\pi}{12} 3h^2 \cdot \frac{dh}{dt}$$

$$= \frac{\pi h^2}{4} \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

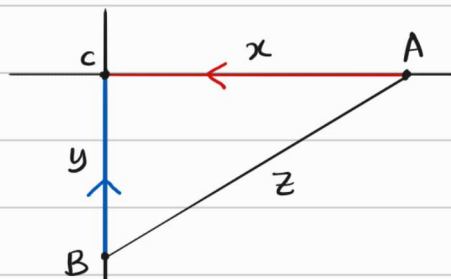
Substituting $h = 3 \text{ m}$ and $V = 2 \text{ m}^3/\text{min}$

$$\frac{dh}{dt} = \frac{4}{\pi (3^2) \text{ m}^2} \cdot 2 \text{ m}^3/\text{min}$$

$$= \frac{8}{9\pi} \text{ m/min.}$$

$$\approx 0.28 \text{ m/min}$$

Eg. Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?



x = distance between car A and the intersection.

y = distance " car B " " "

z = distance between the cars.

Given : $\frac{dx}{dt} = -50 \text{ mi/h}$, $\frac{dy}{dt} = -60 \text{ mi/h}$

(The distances are decreasing).

Unknown : $\frac{dz}{dt}$ when $x = 0.3 \text{ mi}$ and $y = 0.4 \text{ mi}$.

Relation b/w x, y and z :

$$z^2 = x^2 + y^2$$

$$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2 + y^2)$$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \frac{1}{z} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right)$$

when $x = 0.3 \text{ mi}$ and $y = 0.4 \text{ mi}$

we get $z = \sqrt{x^2 + y^2} = \sqrt{(0.3)^2 + (0.4)^2} = 0.5 \text{ mi}$

So, $\frac{dz}{dt} = \frac{1}{0.5} (0.3(-50) + 0.4(-60)) \text{ mi/h}$

$$= -78 \text{ mi/h}$$

The cars are approaching each other at a rate of 78 mi/h.

Exercise: Air is being pumped into a spherical balloon so that its volume increases at a rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm?